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Chords in longest cycles

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If a graph G is 3-connected and has minimum degree at least 4, then some longest cycle in G has a chord. If G is 2-connected and cubic, then every longest cycle in G has a chord.

1. Introduction

In 1976, when I was a graduate student at the University of Waterloo, I raised the question if every longest cycle in a 3-connected graph must have a chord, see [2], [4], [5]. A few years later, when I was convinced that the problem was not trivial, it was published as Conjecture 8.1 in [1] and as Conjecture 6 in [14].

Shortly after my chord-conjecture, Andrew Thomason [13] introduced his elegant and powerful so-called *lollipop method*. About 20 years later, I applied the lollipop method to bipartite graphs [15] and to a weakening of Sheehan's conjecture [17]. Then I realized that the method in [17] had a somewhat unexpected application, namely the chord-conjecture restricted to cubic 3-connected graphs. (For planar cubic 3-connected graphs the conjecture was verified in [19].) Subsequently, the chord-conjecture was verified also

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for other classes of graphs in [10], [11], [9], [3], [18]. As the conjecture is still open, it seems relevant to ask the weaker question: Does every 3-connected graph contain *some* longest cycle which has a chord?

Sheehan's conjecture [12] says that every 4-regular Hamiltonian graph has a second Hamiltonian cycle. Using the lollipop method, it was proved in [17] that there is a second Hamiltonian cycle provided the graph has a red-independent and green-dominating set (where the red edges are the edges of the Hamiltonian cycle and the green edges are the remaining edges). While a 4-regular Hamiltonian graph need not have a red-independent and green-dominating set, it was proved in [17] that such a set exists if the graph is r-regular with $r > 72$. In [8] this was extended to $r > 22$. This idea was carried further in [16] where the chord-conjecture was verified for the class of cubic 3-connected graphs. In that proof a red-independent, green-dominating set (in an appropriate auxiliary graph) was found using the Fleischner–Stiebitz theorem [7] saying that every cycle-plus-triangles graph has chromatic number 3.

The results of the present paper are based on a new application of the lollipop method to cycles containing a prescribed matching in a cubic graph. In the applications we again use the Fleischner–Stiebitz theorem, but we do not use the red-independent, green-dominating sets as we do in [16]. In that paper it is important that the graphs are cubic and 3-connected. The method in this paper also applies to 2-connected cubic graphs.

All graphs in this paper are finite and without loops and multiple edges. The terminology and notation is standard, as [6], [4].

2. Long cycles containing a prescribed matching in a cubic graph

The key idea of the present paper is the following result on long cycles containing a prescribed matching in a cubic graph.

Theorem 1. *Let G be a cubic graph such that $V(G)$ has a partition into sets A, B such that the induced graph $G(A)$ is a matching M, and $G(B)$ is a matching M'. Let $|A| = |B| = 2k$. Assume that G has a cycle C of length $3k$ such that C contains each edge in M, and precisely one end of each edge in M'.*

Then G has a cycle of length $> 3k$ containing M.

Proof of Theorem 1. The proof is by induction on k. For $k = 1$ the statement is trivial, so we proceed to the induction step.

Let the edges of M be denoted $x_1y_1, x_2y_2, \ldots, x_ky_k$, let the edges of M' be denoted $x'_1y'_1, x'_2y'_2, \ldots, x'_ky'_k$, and let $C : x'_1x_1y_1x'_2x_2y_2x'_3 \ldots x_ky_kx'_1$. As in the lollipop argument, we consider an auxiliary graph H. A vertex in H is a path P in G which starts with the edge x'_1x_1, contains all edges of M, has its last edge in M, and if it contains each of x'_i, y'_i, then it also contains the edge $x'_iy'_i$ for $i = 1, 2, \ldots, k$. In particular, P cannot contain the vertex y'_1. Clearly, P contains one or two of x'_i, y'_i for each $i = 1, 2, \ldots, k$. In particular, P has length at least $3k - 1$. Let z be the end in P distinct from x'_1. If z' is

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a neighbor of z in B, then we may assume that $z' \neq y'_1$ since otherwise, there would be a cycle of length at least $3k+1$ containing M. If $z' \neq x'_1$, and if e denotes the unique edge in M' incident with z', then there is a unique path $P' \neq P$ in $P \cup \{zz', e\}$ which is a vertex in the auxiliary graph H. We say that P, P' are neighbors in H. Now, a vertex P in H has degree 1 if its end distinct from x'_1 is a neighbor of x'_1 in G. Otherwise, P has degree 2 in H. As $C - x'_1 y_k$ has degree 1 in H, there is another vertex P' in H which has degree 1 in H. Let C' denote the cycle obtained from P' by adding an edge incident with x'_1. As C' contains M and at least one end of each edge in M', we may assume that C' has length precisely $3k$ and hence C' contains precisely one vertex of each end of each edge in M'.

We color the edges in G as follows: An edge in C but not in C' is blue. An edge in C' but not in C is yellow. An edge in both C and C' is green. An edge in neither C nor C' is black. Note that every edge in M is green, and also $x'_1 x_1$ is green. Since $C' \neq C$, it follows that some edges are blue, and some edges are yellow. Every edge $x'_i y'_i$ in M' is black. The other two edges incident with x'_i (respectively y'_i) have the same color, say $c(x'_i)$ (respectively $c(y'_i)$). The two colors $c(x'_i), c(y'_i)$ are either black, green or blue, yellow. Now consider a maximal green path Q. It starts and ends with an edge in M because of the above observations on the colors $c(x'_i), c(y'_i)$. All four edges joining the ends of Q to ends of M' are blue or yellow by the maximality of Q. All other edges incident with Q are black. We now delete all those vertices in G which are incident with three black edges. In the resulting graph we suppress all vertices of degree 2, that is, we replace each path with endvertices of degree 3 and intermediate vertices of degree 2 by a single edge. This results in a cubic graph G_1. The maximal green paths in G become a green matching M_1 with k_1 edges, say, in G_1. Since $x'_1 x_1$ is green, we have $k_1 < k$. The black edges that have not been deleted form a matching M'_1. Now the cycle C in G corresponds to a cycle C_1 in G_1 containing M_1 and precisely one end of each edge in M'_1. By the induction hypothesis, G_1 contains a cycle of length $> 3k_1$ containing M_1. This corresponds to a cycle of length $> 3k$ in G. $\square$

3. Chords in longest cycles in cubic 2-connected graphs

We first establish a variation of Thomason's lollipop theorem.

Theorem 2. *Let G be a connected graph such that no two vertices of even degree are joined by an edge. Let C be a cycle in G such that all vertices in $G - V(C)$ have even degree. Then G has a cycle C' distinct from C such that C' contains all vertices of odd degree.*

Proof of Theorem 2. We may assume that no vertex in $G - V(C)$ is joined to two consecutive vertices of C since otherwise, there exists a cycle containing $V(C)$ and one more vertex. Let $C : v_1 v_2 \dots v_n v_1$ such that v_n has odd degree. As in the lollipop argument, we consider an auxiliary graph H. A vertex in H is a path P in G which starts with

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the edge v_1v_2, contains all vertices of odd degree, and ends with a vertex of odd degree. Consider such a path P whose end distinct from v_1 is denoted z. Consider an edge zy or a path zuy where y is in $P - v_1$ and u is in $G - V(P)$. If we add the edge zy or the path zuy to P and then delete the vertex succeeding y on P (if that vertex has even degree in G) or delete just the edge succeeding y on P otherwise, then the resulting path P' is a vertex of H. We say that P, P' are neighbors in H. If there is no edge between z, v_1 and there is no path zuv_1 with u being a vertex in $G - V(P)$, then clearly P has even degree in H. The path $C - v_1v_n = v_1v_2 \dots v_n$ clearly has odd degree in H because there is no path v_nuv_1 with u being a vertex of $G - V(C)$. But then there is another vertex Q, say, of odd degree in H. If Q ends at z, and z, v_1 are neighbors, then $Q \cup \{zv_1\}$ is a cycle distinct from C containing all vertices of odd degree. If there is a path zuv_1 where u is a vertex in $G - V(Q)$, then the union of Q and the path zuv_1 is a cycle containing all vertices of odd degree. This cycle is distinct from C because u, the predecessor of v_1, has even degree. $\square$

Theorem 3. *Every longest cycle in a 2-connected cubic graph has a chord.*

Proof of Theorem 3. Let G be a 2-connected cubic graph. Let C be a longest cycle in G. Assume (reductio ad absurdum) that C has no chord. We form a new graph G_1 as follows: If H is a connected component of $G - V(C)$ joined to at least three vertices of C, then we contract H to a single vertex which we call a *pleasant vertex*. In particular, every component of $G - V(C)$ with precisely one vertex is a pleasant vertex. If H is joined to only two vertices x, y of C, then we replace H by an edge xy. This edge is called a *pleasant edge*. For each pleasant vertex in G_1 we select three neighbors on C called *pleasant neighbors* of the pleasant vertex. For each pleasant vertex we call one of its pleasant neighbors *very pleasant*. By the Fleischner–Stiebitz theorem [7] we can select the very pleasant neighbors in such a way that no two of them are consecutive on C. To see that we form a so-called cycle-plus-triangles graph from the cycle C by adding a triangle consisting of the three pleasant neighbors of each pleasant vertex. The Fleischner–Stiebitz theorem implies that this graph is 3-colorable, and we now let the very pleasant neighbors be the pleasant neighbors of color 1, say.

So far the present proof is similar to the proof in [16]. The proof in [16] then uses the method in [17]. However, this does not work if there are pleasant edges. Therefore the graphs in [16] are assumed to be 3-connected. Here we instead first use Theorem 2 and then Theorem 1.

A cycle C_1 in G_1 is called *pleasant* if it contains all vertices of C except possibly some very pleasant neighbors. We shall now investigate a cycle C_1 which is pleasant in G_1 and distinct from C. Let r be the number of vertices in C but not in C_1. Let p, q be the number of pleasant vertices and pleasant edges, respectively, in C_1. Clearly C_1 can be transformed to a cycle in G by adding a path in each component of $G - V(C)$ which corresponds to a pleasant vertex or edge contained in C_1. With a slight abuse of notation we denote this cycle in G by C_1. In this way a pleasant edge in C_1 corresponds to a path

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with at least 3 edges in G. (In fact that path can be chosen such that it has at least 5 edges but we shall not need that.) So, the cycle C_1 in G is at least as long as the cycle C_1 in G_1, and if C_1 in G_1 contains a pleasant edge, then C_1 in G is strictly longer. We claim that the length of C_1 in G_1 is at least (and hence equal to) the length of C in G.

To prove this claim we focus on C_1 in G_1. Suppose x is one of the very pleasant neighbors not contained in C_1. Then C_1 contains both neighbors of x on C. Let y be one of those two neighbors. Then C_1 contains a pleasant edge yz or a path yuz where u is a pleasant vertex. We say that the edge yz or the vertex u *dominates* x. Possibly, yz or u also dominates a neighbor of z on C. The other neighbor y' of x on C is also incident with a pleasant edge $y'z'$ or path $y'u'z'$, and we say that the edge $y'z'$ or vertex u' also dominates x. So there are precisely two elements dominating x. Since a pleasant vertex or edge dominates at most two vertices, it follows that $p + q \geq r$.

The number of edges in C_1 in G_1 is $|E(C)| + 2p + q - 2r$. The length of C_1 in G is at least $|E(C)| + 2p + 3q - 2r$. As C is longest in G, it follows that $q = 0$ and $p = r$. In other words, C' contains no pleasant edge and has the same edges in G as in G_1, and each vertex in $C_1 - V(C)$ dominates precisely two vertices.

We now describe a new graph G_2 from G_1. If u is a pleasant vertex in G_1, and u' is its very pleasant neighbor, then we contract the edge uu' into a vertex which we also call u'. We apply Theorem 2 to the graph G_2. The resulting cycle distinct from C is called C_2. The edge set of the cycle C_2 can be extended to the edge set of a cycle C_1 in G_1 by possibly adding some of the contracted edges of the form uu'. Clearly, C_1 is pleasant in G_1. This implies that C_1 contains no edge of the form uu' where u is pleasant and u' is a very pleasant neighbor because in that case u would not dominate a neighbor of u' on C, and we know that u dominates two vertices. So C_2, C_1 have the same edge set. If C_1 contains the pleasant vertex u, then C' does not contain its very pleasant neighbor u'. Since $p = r$, the converse holds: If C' does not contain the very pleasant neighbor u' of u, then C_1 contains u.

Now let Q denote the graph which is the union of C and C_1 and all edges of the form uu' where u is a pleasant vertex in C_1 and u' is its very pleasant neighbor in C. These edges form a matching M'. Let Q' be obtained from Q by suppressing all vertices of degree 2. The maximal paths that C and C_1 have in common each has length > 0 (because G is cubic) and hence these paths form a matching M in Q'. We now apply Theorem 1 to Q'. By Theorem 1, Q' has a cycle which contains all edges in M and which is longer that C. Then also G has such a longer cycle, a contradiction which proves Theorem 3. □

4. Chords in longest cycles in 3-connected graphs of minimum degree at least 4

If x is a vertex in a graph G, we call the degree of x in G the *G-degree*. The following lemma is a well-known exercise.

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Lemma 1. *If A is an even vertex set in a connected G, then G has a spanning subgraph H such that every vertex in A has odd H-degree, and all other vertices have even H-degree.* □

Proposition 1. *Let C be a chordless cycle in a graph G of minimum degree at least 3 such that the vertices in $G - V(C)$ form an independent set (that is, they are pairwise nonadjacent). Then G has a cycle C' such that either C' is longer than C, or C' has the same length as C and has a chord.*

Moreover, if G is minimal in the sense that every edge in $G - E(C)$ is incident with a vertex of G-degree 3, then C' can be chosen such that it has a chord incident with a vertex in $G - V(C)$ which has G-degree 3.

Proof of Proposition 1. Assume without loss of generality that G is edge-minimal, that is, if we delete an edge in $G - E(C)$ or a vertex in $G - V(C)$, then we create a vertex of degree 2 in the resulting graph. So, if v is a vertex in $G - V(C)$, then v has a neighbor on C of degree 3. If v has degree at least 4, then all neighbors of v have degree 3. For every component Q in $G - E(C)$ we select three vertices x_Q, y_Q, z_Q in $V(Q) \cap V(C)$ such that as many as possible have degree 3 in G. It is easy to see that all of x_Q, y_Q, z_Q have degree 3 unless Q has 6 vertices x_Q, y_Q, z_Q, u, v, w such that x_Q, y_Q, z_Q, w are in C, u, v are outside C, u is joined to x_Q, y_Q, w, and v is joined to z_Q, y_Q, w. We now apply the Fleischner–Stiebitz theorem [7] to the cycle-plus-triangles graph obtained from C by adding the three edges $x_Q y_Q, x_Q z_Q, y_Q z_Q$ for each component Q of $G - E(C)$. The resulting graph is 3-chromatic. We rename vertices such that all the vertices of the form x_Q have the same color. In particular, these vertices are independent. Now consider a component Q of $G - E(C)$. If Q has only one vertex u_Q outside C we contract the edge $u_Q x_Q$. If Q has more than one vertex outside C (and hence all vertices outside C have G-degree precisely 3), then we let Q' be a spanning subgraph of Q such that all vertices in $V(C) \cap V(Q)$ (except possibly x_Q) have odd Q'-degree and all other vertices in Q' have even Q'-degree. If all vertices in $V(C) \cap V(Q)$ have odd Q'-degree, then we delete from G all edges in $E(Q) \setminus E(Q')$. If x_Q has even Q'-degree, then Q is not the afore-mentioned component with 6 vertices (because that component has an even number of vertices in C), and hence x_Q has a unique neighbor u_Q in Q and has Q'-degree 0. We contract the edge between x_Q and u_Q and we delete from G all other edges in $E(Q) \setminus E(Q')$. We call the resulting graph G', and we apply Theorem 2 to G'. Let C'' be a cycle distinct from C and containing all vertices in C which have odd G'-degree. Let C' be the corresponding cycle in G. We now investigate C' in the same way as we investigated C_1 in the proof of Theorem 3. As pointed out by a referee, there may be a path $x_1 u x_2$ in C and a path $y_1 u y_2$ in C' such that x_1, x_2 are outside C' and y_1, y_2 are outside C, a situation that does not occur in Theorem 3. In that case we replace u by two vertices u_1, u_2 and replace the paths $x_1 u x_2$ and $y_1 u y_2$ by $x_1 u_1 u_2 x_2$ and $y_1 u_1 u_2 y_2$, respectively. With a slight abuse of notation we still use G, C, C' for the modified graphs. Then every vertex in $C \cup C'$ has degree at most 3 which allows us to use Theorem 1 as shown below. Let r be the number

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of vertices in C but not in C'. Let p be the number of vertices in C' but not in C. As in the proof of Theorem 3 we conclude that $p \geq r$. If $p > r$, then C' is longer than C, so assume that $p = r$. Consider one of the p vertices in $C' - V(C)$, say u. If each such u has a neighbor on C which is not in C', then, as in the proof of Theorem 3, we use Theorem 1 to conclude that G has a cycle which is longer than C. One the other hand, if some such u has the property that each of its neighbors on C is also in C', then no neighbor of u is of the form x_Q. Then u has G-degree 3, and one of its three incident edges is a chord in C'. This proves Proposition 1. $\square$

Corollary 1. *Let C be a longest cycle in a 3-connected graph G. If C is chordless, then G has a longest cycle C' distinct from C.*

Proof of Corollary 1. Contract each component of $G - V(C)$ into a vertex. Then C is a longest cycle in the resulting graph. Now apply Proposition 1. $\square$

Theorem 4. *Let C be a chordless cycle in a 3-connected graph G of minimum degree at least 4. Then G has a cycle C' such that either C' is longer than C, or C' has the same length as C and has a chord.*

Proof of Theorem 4. The idea in the proof is to contract each component of $G - V(C)$ into a single vertex and then apply the method of Proposition 1. The problem is that a chord in the resulting graph need not be a chord in G in case the new cycle contains some of the contracted vertices. For example, the two edges in the new cycle incident with the contracted vertex v' may also be incident with the same vertex v in G, and the chord may be incident with v' but not with v.

To deal with that problem we need a technical investigation of the components of $G - V(C)$.

We may assume that some component of $G - V(C)$ has at least two vertices since otherwise, Theorem 4 follows from Proposition 1.

If a component of $G - V(C)$ has precisely two vertices, we delete the edge between them. (This is the only place where we use that vertices outside C have degree at least 4.) Note that each of these vertices has at least three neighbors on C. With a slight abuse of notation we also call the resulting graph G. If a component Q in $G - V(C)$ has more than one vertex, then it now has at least three vertices and hence the edges between Q and C contain a matching with at least 3 edges.

We shall delete edges between C and $G - V(C)$ in order to obtain a spanning subgraph G' of (the new) G such that each vertex of C has G'-degree at least 3 and such that, for each component Q in $G - V(C)$ with more than one vertex, the edges in G' between Q and C contain a matching with at least 3 edges.

We say that a component Q in $G' - V(C) = G - V(C)$ satisfying at least one of (i), (ii), (iii) below is a *good component*.

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(i) Q has only one vertex, and there are precisely 3 edges between Q and C.
(ii) There are precisely 3 edges between Q and C, and they form a matching.
(iii) Q has at least 3 neighbors on C of G'-degree precisely 3, and, if Q has more than one vertex, then G' has a matching with 3 edges between Q and C.

We choose G' such that the number of non-good components is minimum, and subject to this G' has as few edges as possible between C and $G - V(C)$.

We define a *bad component* of $G' - V(C)$ as a component Q satisfying each of (iv), (v), (vi), (vii) below, where

(iv) there are precisely 4 edges between Q and C.
(v) Precisely two of them, say $z_Q x_Q, z_Q y_Q$ have an end z_Q in common, and that end is in Q.
(vi) x_Q, y_Q each has G'-degree precisely 3.
(vii) The two neighbors of Q on C distinct from x_Q, y_Q each has G'-degree > 3.

Clearly, a bad component is not good. We shall prove that every non-good component is bad.

If a component of $G - V(C)$ has precisely one vertex, and it has G'-degree > 3, then each neighbor has G'-degree precisely 3, since otherwise we can delete an edge and contradict the minimality of G'. So, a component of $G - V(C)$ with precisely one vertex satisfies (i) or (iii). If a component Q in $G - V(C)$ has more than one vertex, then it has at least three vertices and hence the edges between Q and C contain a matching with at least 3 edges. Consider a maximum matching M between Q and C. Then M has at least 3 edges. If M has more than 3 edges, then each end of M in C has G'-degree 3, by the minimality of G', and hence Q satisfies (iii). So assume that M has precisely 3 edges q_1c_1, q_2c_2, q_3c_3 where q_1, q_2, q_3 are in Q. If the edges of M are the only edges from Q to C, then (ii) holds. So assume there are more edges from Q to C. Each edge from Q to C not in M joins one of q_1, q_2, q_3 with a vertex in C distinct from c_1, c_2, c_3 and of G'-degree 3, by the minimality of G'. Consider such an edge q_1c_4. Then c_4 has degree 3. Since q_1c_4, q_2c_2, q_3c_3 is also a matching, c_1 has degree 3. If one (or both) of q_2, q_3 is joined to more than one vertex of C, then Q has at least three neighbors on C of degree precisely 3, and then Q satisfies (iii). So assume q_2, q_3 each have only one neighbor on C. If one or both of c_2, c_3 has degree 3, then again, Q satisfies (iii). So, both of c_2, c_3 have degree > 3. Hence Q is bad.

This discussion proves:

Claim 1. If a component Q of $G' - V(C)$ is not good, then it is bad.

Next we prove that all components of $G' - V(C)$ are good.

Consider therefore a bad component Q in $G' - V(C)$. Recall that Q has a vertex z_Q with G'-neighbors x_Q, y_Q of Q'-degree precisely 3. But, they have G-degree at least 4. (This is the only place where we use that vertices in C have G-degree at least 4.) Let x

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be a neighbor of x_Q not in C and distinct from z_Q. If x is in Q, then we add to G' the edge x_Qx and delete the edge z_Qx_Q and one more edge from Q to C so that the resulting graph has fewer edges than G' and the new Q satisfies (ii) and is therefore good. So we may assume that x is in a component $Q_1 \neq Q$. If we add x_Qx and delete x_Qz_Q, then Q changes from bad to good. The minimality property of G' implies that Q_1 changes from good to not good and hence, by Claim 1, to bad. In other words, the vertex x is the unique vertex of Q_1 with a G'-neighbor x' in C of G'-degree 3. If $q > 1$ we obtain a contradiction by adding the red edges to Q_q, Q and deleting an edge from Q_q to C. So assume we must have $q = 1$. We may assume that, for every bad component Q, there is a component Q_1 satisfying (ii) such that there are red edges z_Qx', x_Qx not in G' and there is an edge xx' in G' where x' is the unique neighbor of Q_1 with G'-degree precisely 3. We call Q, Q_1 a *good pair*. If there is a good pair Q', Q_1 where Q' is distinct from Q, we easily get a contradiction by making Q, Q' satisfy (ii) and Q_1 satisfy (iii). We now consider all good pairs one by one. We add the red edge from z_Q to C and delete all vertices of $Q - z_Q$. We also delete Q_1. We repeat this for any other good pair. (Note that some good pair may no longer be a good pair after the deletion of Q_1 and $Q - z_Q$. In that case we can reduce the number of bad components as above.) This shows that we may assume:

Claim 2. If Q is a component of $G' - V(C)$, then Q is good.

We now delete edges from the components Q satisfying (iii) to C such that all vertices on C still have degree at least 3, and the following weaker statement (iii)′ is satisfied, where

(iii)′ Q has at least 3 neighbors on C, and all neighbors of Q on C have degree precisely 3.

With a slight abuse of notation we call the resulting graph G'.

Now we contract each component Q of $G' - V(C)$ into a vertex w_Q. We call the resulting graph H. Now we repeat the proof of Proposition 1 with H instead of G. As in the proof of Proposition 1 we assume that H is edge-minimal, that is, each vertex w_Q has a vertex on C of H-degree 3, and if w_Q has H-degree > 3, then all neighbors on C have H-degree 3. Let C' be the cycle of the same length as C obtained in the proof of Proposition 1. We may assume that G has no cycle of length greater than the length of C. Hence C' contains a vertex $u = w_Q$ of H-degree 3 which is not in C and which has the property that each of its neighbors on C is also in C'. So, C' has a chord incident with $u = w_Q$. As the edge set of C' can be extended to a cycle in G, and since C is a longest cycle in G we conclude that the edges of C' form a cycle in G. We claim that the chord of C' in H is also a chord of C' in G. To see this we first observe that no neighbor of u is a vertex of the form x_Q found in the proof of Proposition 1 by the Fleischner–Stiebitz theorem (since u and that vertex x_Q would have been identified before we used Theorem 2 in the proof of Proposition 1). (Note that the Q in x_Q in Proposition 1 has a slightly different meaning than in the present proof.) So Q does not satisfy (iii)′. Secondly, Q cannot satisfy (ii) because the edges of C' form a cycle in G. As Q satisfies (i) or (ii) or (iii)′, by the choice of G', it follows that Q satisfies (i). Hence the chord of C' in H is also a chord of C' in G.

This proves Theorem 4. □

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